

Asymptotics of alternating harmonic series with attenuation

Sergey Sadov*

Abstract

We find the asymptotics of the series $\sum_{n=1}^{\infty} (-1)^n n^{-1} \exp(-t/n)$ as $t \rightarrow +\infty$. The answer is an oscillating function of t dominated by $\exp(-(2\pi t)^{1/2})$. The intermediate step is to find the asymptotics of the two-dimensional Fourier transform $\hat{F}(\xi)$ of the function $F(x) = (1 + \exp(\|x\|^2))^{-1}$ as $\|\xi\| \rightarrow \infty$.

Keywords: Asymptotics, harmonic series, model problem, Bessel functions, Hankel transform, Fourier transform

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The problem discussed in this note is an yet another example of a challenge born of a teaching mishap. I offered it by mistake among a set of exercises on the Euler-Maclaurin formula in a course of asymptotic analysis at Memorial University of Newfoundland in the Fall 2011.

Problem. Find asymptotics of the series

$$S(t) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-t/n} \quad (1)$$

as $t \rightarrow +\infty$.

In the interests of those who want to take up the challenge the answer is only given at the end of the note. The solution below is long but detailed. It is intended to be understood by an asymptotic analysis course student who wants to get their hand dirty with the Saddle Point Method.

*E-mail: serge.sadov@gmail.com

Solution.

Part 1: Derivation of integral representation

Let us consider a more general series

$$S(z, \nu, t) = \sum_{n=1}^{\infty} \frac{z^n}{n^\nu} e^{-t/n}. \quad (2)$$

Lemma 1 (a) *The series (2) converges absolutely if $|z| < 1$ for any $\nu \in \mathbb{R}$.*

(b) *If $\nu > 1$, then the series converges absolutely and uniformly in the closed unit circle $|z| \leq 1$.*

(c) *If $0 < \nu \leq 1$, $|z| = 1$, $z \neq 1$, the series converges conditionally. Also, for any $\varepsilon \in (0, 2)$, the convergence is uniform in the region*

$$D_\varepsilon = \{z : |z| \leq 1, \operatorname{Re} z \leq 1 - \varepsilon\}.$$

(d) *Consequently, if $|z| = 1$, $z \neq 1$, then*

$$S(z, \nu, t) = \lim_{\rho \rightarrow 1^-} S(z\rho, \nu, t). \quad (3)$$

Proof. Parts (a) and (b) are obvious since the series is dominated by $\sum |z|^n n^{-\nu}$ in the case (a) and by the z -independent sum $\sum n^{-\nu}$ in the case (b).

To prove (c), let us write $e^{-t/n} = 1 - \delta_n$, where $\delta_n = O(n^{-1})$ (for a fixed t). The series $\sum z^n n^{-\nu} \delta_n$ converges absolutely and uniformly in $\{|z| \leq 1\}$ (similarly to the series $S(z, \nu + 1, t)$). It remains to establish the convergence properties (c) for the series ¹

$$\sum_{n=1}^{\infty} n^{-\nu} z^n.$$

It is a standard application of Dirichlet's test: the sequence $\{n^{-\nu}\}$ decreases and the uniform in D_ε bound for the partial sums

$$\left| \sum_{n=1}^N z^n \right| \leq \frac{2}{|1 - z|} \leq \frac{2}{\varepsilon}$$

holds.

¹The function $\operatorname{Li}_\nu(z)$ defined by this series is called *polylogarithm of order ν* .

To prove (d), write $z = e^{i\theta}$, where $\theta \in (0, 2\pi)$. Let $\varepsilon > 0$ be such that $\varepsilon < 1 - \cos \theta$. Then $\rho z \in D_\varepsilon$ for any $\rho \in [0, 1]$. The series

$$\sum_{n=1}^{\infty} \frac{(\rho e^{i\theta})^n}{n^\nu} e^{-t/n}$$

converges uniformly w.r.to ρ . Hence termwise passing to the limit as $\rho \rightarrow 1^-$ is justified, and (3) follows. \square

Remark. In Lemma 1, the parameter t can be any complex number. For fixed z ($|z| \leq 1$, $z \neq 1$) and $\nu > 0$, the function $S(z, \nu, t)$ is an entire analytic function of t .

It is interesting also to note the relations

$$\begin{aligned} \frac{\partial S(z, \nu, t)}{\partial t} &= -S(z, \nu + 1, t), \\ \frac{\partial S(z, \nu + 1, t)}{\partial z} &= z^{-1} S(z, \nu, t), \\ \frac{\partial^2 S(z, \nu, t)}{\partial t \partial z} &= -z^{-1} S(z, \nu, t). \end{aligned}$$

\square

Recall the series representing Bessel function of order $n \geq 0$:

$$J_n(u) = \sum_{k=0}^{\infty} \frac{(-1)^k (u/2)^{2k+n}}{k! \Gamma(k+n+1)}. \quad (4)$$

Lemma 2 *For $\nu \geq 1$ and $|z| < 1$, the function $S(z, \nu, t)$ has the integral representation*

$$S(z, \nu, t) = \int_0^\infty \frac{ze^{-x}}{1 - ze^{-x}} \left(\frac{x}{t}\right)^{\frac{\nu-1}{2}} J_{\nu-1}(2\sqrt{tx}) dx. \quad (5)$$

The same representation remains valid if $|z| = 1$, $z \neq 1$. In particular,

$$S(t) = S(-1, 1, t) = - \int_0^\infty \frac{1}{e^x + 1} J_0(2\sqrt{tx}) dx. \quad (6)$$

Proof. Suppose $|z| < 1$. Since

$$\frac{\Gamma(k+\nu)}{n^{k+\nu}} = \int_0^\infty e^{-nx} x^{k+\nu-1} dx,$$

we have

$$\begin{aligned}
S(z, \nu, t) &= \sum_{n=1}^{\infty} z^n \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} n^{-\nu-k} \\
&= \sum_{n=1}^{\infty} z^n \sum_{k=0}^{\infty} \frac{(-t)^k}{k! \Gamma(k+\nu)} \int_0^{\infty} e^{-nx} x^{k+\nu-1} dx \\
&= \sum_{k=0}^{\infty} \frac{(-t)^k}{k! \Gamma(k+\nu)} \int_0^{\infty} \left(\sum_{n=1}^{\infty} z^n e^{-nx} \right) x^{k+\nu-1} dx \\
&= \sum_{k=0}^{\infty} \frac{(-t)^k}{k! \Gamma(k+\nu)} \int_0^{\infty} \frac{ze^{-x}}{1-ze^{-x}} x^{k+\nu-1} dx \\
&= \int_0^{\infty} \frac{ze^{-x}}{1-ze^{-x}} x^{\nu-1} \sum_{k=0}^{\infty} \frac{(-tx)^k}{k! \Gamma(k+\nu)} dx.
\end{aligned}$$

In the above transformations, convergence is absolute throughout, so we can interchange the order of summations and integration freely.

Setting $u = 2\sqrt{tx}$ and comparing the integrand in the RHS with (4), we see that

$$x^{\nu-1} \sum_{k=0}^{\infty} \frac{(-tx)^k}{k! \Gamma(k+\nu)} = \left(\frac{x}{t}\right)^{\frac{\nu-1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k (u/2)^{2k+\nu-1}}{k! \Gamma(k+\nu)} = \left(\frac{x}{t}\right)^{\frac{\nu-1}{2}} J_{\nu-1}(u),$$

so that (5) follows.

Denote temporarily the RHS of (5) by $S_*(z, \nu, t)$. The integral in (5) converges absolutely if $|z| \leq 1$, $z \neq 1$. (Convergence at $x \rightarrow \infty$ takes place for any z and convergence at $x \rightarrow 0^+$ is certain if $\nu \geq 1$, $z \notin [1, \infty)$.) Consequently, if $|z| = 1$, $z \neq 1$,

$$\lim_{\rho \rightarrow 1^-} S_*(z\rho, \nu, t) = S_*(z, \nu, t).$$

On the other hand, Lemma 1 asserts the same for $S(z, \nu, t)$. Thus, (5) remains valid in the case $|z| = 1$, $z \neq 1$ by continuity. \square

Corollary. Let

$$\lambda = 2\sqrt{t} \quad \text{and} \quad S(t) = S_*(\lambda). \quad (7)$$

Taking \sqrt{x} as the new variable of integration in (6), we get ²

$$S_*(\lambda) = - \int_0^{\infty} J_0(\lambda x) \frac{2x}{e^{x^2} + 1} dx. \quad (8)$$

² $S_*(\lambda)$ is called the *Hankel transform* of the function $-2x/(\exp(x^2) + 1)$.

Part 2: Transformations of the integral (8)

Substituting the integral representation

$$J_0(u) = \frac{1}{2\pi} \int_0^{2\pi} e^{iu \cos \theta} d\theta,$$

into (8) we express $S_*(\lambda)$ as a double integral

$$S_*(\lambda) = -\frac{1}{2\pi} \int_0^\infty \int_0^{2\pi} e^{i\lambda x \cos \theta} d\theta \frac{2x}{e^{x^2} + 1} dx. \quad (9)$$

Experimental study suggests that $S_*(\lambda) = O(e^{-\lambda})$. Unfortunately, it is not clear how a precise asymptotics can be derived from the representation (9), because the edge $x = 0$ interferes in all attempts: integration by parts or rotation of the integration path in the complex x -plane. We will not describe such attempts in detail, since the successful solution will be based on a different transformation of the integral (8). Note only that if instead of the factor $2x$ we had an even function of x in the integrand of (9), then it would be possible to extend domain of integration w.r.t. x to \mathbb{R} , getting rid of the edge; the saddle point method would then be applicable in a standard manner.

The representation we need will come from a known general formula of Fourier analysis.

Lemma 3 Suppose $\int_0^\infty |f(r)|r dr < \infty$. Let $r = r(x, y) = (x^2 + y^2)^{1/2}$. Then the 2-dimensional Fourier transform of $F(x, y) = f(r(x, y))$ is

$$\hat{F}(\xi, \eta) \equiv \int_{\mathbb{R}^2} F(x, y) e^{-ix\xi - iy\eta} dx dy = 2\pi \int_0^\infty J_0(r\rho) f(r) r dr, \quad (10)$$

where $\rho = (\xi^2 + \eta^2)^{1/2}$.

Proof. Using the polar coordinates (r, θ) in the (x, y) plane and (ρ, ϕ) in the (ξ, η) plane, we get

$$\hat{F}(\xi, \eta) = \int_0^\infty \int_0^{2\pi} f(r) e^{-ir\rho \cos(\phi - \theta)} d\theta r dr.$$

Clearly, the change of variable $\theta \rightarrow \theta - \phi$ shows that the integral does not depend on ϕ ; thus

$$\begin{aligned} \hat{F}(\xi, \eta) &= \int_0^\infty f(r) r \int_0^{2\pi} f(r) e^{ir\rho\theta} d\theta dr \\ &= \int_0^\infty f(r) r \cdot 2\pi J_0(r\rho) dr, \end{aligned}$$

as stated. □

Corollary. The integral (8) can be written as the 2-dimensional Fourier integral

$$S_*(\lambda) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{1 + e^{x^2+y^2}} dx dy. \quad (11)$$

Indeed, it suffices to replace r in (8) by r and to use (10) with $f(r) = 2/(e^{r^2+1} + 1)$, $(\xi, \eta) = (-\lambda, 0)$.

Remark. Expanding $(1 + e^{x^2+y^2})^{-1}$ as a geometric series and integrating separately w.r.t. x and y , one obtains an independent proof of the fact that $S(t)$ equals the RHS of (11) when $\lambda = 2\sqrt{t}$.

Part 3: Asymptotic analysis of the integral (11)

Asymptotics of the integral (11) is determined by the complex singularities of the integrand. As a preparation, let us locate complex zeros in the z -plane of the denominator

$$Q(z, y) = 1 + e^{z^2+y^2}.$$

where $z = x + iu$, and x, u, y are real.

The equation $Q(z, y) = 0$ is equivalent to

$$z^2 + y^2 = i(2k + 1)\pi, \quad k \in \mathbb{Z}.$$

Or, in the real form,

$$\begin{aligned} x^2 + y^2 &= u^2, \\ xu &= \pi \left(k + \frac{1}{2} \right). \end{aligned} \quad (12)$$

Precise asymptotic analysis of the integral (11) is significantly more complicated than singularities-based analysis of Fourier integrals in the one-dimensional case. As a first, rather easy step, we obtain a rough exponential o -estimate.

Lemma 4 *For any $a < \sqrt{\pi/2}$ we have*

$$S_*(\lambda) = o(e^{-a\lambda}). \quad (13)$$

Proof. Clearly, for any fixed $y > 0$ and $h > 0$

$$|1 + e^{(x+ih)^2+y^2}| \rightarrow \infty$$

as $x \rightarrow \pm\infty$. Therefore, assuming $Q(x+ih, y) \neq 0, \forall x \in \mathbb{R}$, we have

$$\int_{-\infty}^{\infty} \frac{e^{i\lambda x} dx}{Q(x, y)} = \int_{-\infty}^{\infty} \frac{e^{i\lambda(x+ih)} dx}{Q(x+ih, y)} + 2\pi i \sum \text{Res} \frac{e^{i\lambda z}}{Q(z, y)}, \quad (14)$$

where the sum of residues is taken over all $z = x + iu$ such that $0 < u < h$ and $Q(z, y) = 0$.

The equation $Q(x + iu, y) = 0$ does not have solutions with $x, y \in \mathbb{R}$, $0 \leq u < \sqrt{\pi/2}$. Indeed, looking at the system (12), we see that $|x| \leq u$ and $|\pi/2| \geq |xu| \geq u^2$.

Therefore, for $h < \sqrt{\pi/2}$, the sum in the RHS of (14) is void. Thus

$$S_*(\lambda) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{i\lambda(x+ih)}}{1 + e^{(x+ih)^2+y^2}} dx dy = O(e^{-\lambda h}).$$

Given $a \in (0, \sqrt{\pi/2})$, the estimate (13) follows by taking $h \in (a, \sqrt{\pi/2})$.

□

We want eventually to nail down an asymptotical term of the exponential order $O(e^{-\lambda/\sqrt{2}})$ precisely.

The procedure consists of simple steps, but it is rather delicate overall. We need to take into consideration some poles of $Q(z, y)^{-1}$ to obtain a nonempty the residue sum in the RHS (14). On the other hand, we must avoid stepping on a pole during the double integration (in both x and y). Moreover, any estimates we get while keeping y fixed should be uniform or explicit enough to justify subsequent integration with respect to y .

The following lemma will allow us to control the integrand in the double integral uniformly.

Lemma 5 *Suppose $u > 0$ and a closed set $K \subset \mathbb{R}$ are such that the equation $Q(x + iu, y) = 0$ does not have solutions $x \in \mathbb{R}, y \in K$. Then there exists $\alpha = \alpha(a, K) > 0$ such that*

$$|Q(x, y)| \geq \alpha e^{x^2+y^2} \quad \forall x \in \mathbb{R}, y \in K. \quad (15)$$

Proof. We will estimate $|Q(x, y)|$ from below in two cases separately: first for large $x^2 + y^2$, then in a bounded region of the (x, y) plane.

1) Suppose that $x^2 + y^2 > u^2 + \ln 2$. Then $2e^{u^2} < e^{x^2+y^2}$, hence

$$\begin{aligned} |Q(x, y)| &\geq |e^{\operatorname{Re}(x+iu)^2+y^2} - 1| = e^{-u^2}(e^{x^2+y^2} - e^{u^2}) \\ &\geq \frac{e^{-u^2}}{2} e^{x^2+y^2}. \end{aligned}$$

2) The set

$$\Omega = \{(x, y) \mid x^2 + y^2 \leq u^2 + \ln 2, y \in K\}$$

is a bounded closed subset of \mathbb{R}^2 , hence compact. By assumption, $Q(x, y) \neq 0$ when $(x, y) \in \Omega$. The function $e^{x^2+y^2}/Q(x, y)$ is continuous in Ω , hence bounded: $|e^{x^2+y^2}/Q(x, y)| \leq C$.

The inequality (15) with

$$\alpha = \min \left\{ \frac{e^{-u^2}}{2}, \frac{1}{C} \right\}$$

follows. \square

Next comes the crucial step. We choose a splitting parameter b for integration with respect to y and write (11) in the form

$$-\pi S_*(\lambda) = \int_{|y|>b} T(y, \lambda) dy + \int_{|y|\leq b} T(y, \lambda) dy = I_1 + I_2, \quad (16)$$

where

$$T(y, \lambda) = \int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{1 + e^{x^2+y^2}} dx. \quad (17)$$

We are going to make a complex shift $x \rightarrow x + ia$ in the integral (17). However, we will use two different values of a in the two cases, $|y| \geq b$ and $|y| \leq b$.

In the case $|y| \geq b$, we will take $a = a_1$ so as to ensure that no zeros of $Q(z, y)$ lie in the strip $0 \leq \operatorname{Im} z \leq a_1$. It will be possible to choose $a_1 > \sqrt{\pi/2}$ and to obtain the estimate $I_1 = O(e^{-\lambda a_1})$.

In the case $|y| \leq b$, we will take $a = a_2 > \sqrt{\pi/2}$ to ensure that the equation $Q(z, y) = 0$ has exactly two solutions $z_{\pm} = z_{\pm}(y)$ (with the same imaginary part) in the strip $0 \leq \operatorname{Im} z \leq a_1$. The principal term of the asymptotics of the integral (11) will come from the residue part of the RHS in (14).

The three parameters b , a_1 and a_2 in the outlined program are not determined rigidly and can be varied as long as we don't cross certain boundaries. Details are clarified in the following two lemmas.

Lemma 6 *For any a satisfying the inequalities*

$$\sqrt{\frac{\pi}{2}} < a < \sqrt{\frac{3\pi}{2}},$$

there exists a unique $b > 0$ such that

(i) *the equation $Q(x + iu, y) = 0$ does not have real solutions (x, u, y) with $0 \leq u \leq a$ and $|y| > b$;*

(ii) *for every $y \in (-b, b)$, the system*

$$Q(x + iu, y) = 0, \quad 0 \leq u < a$$

has exactly two real solutions (x, u) . They have the same imaginary part and opposite real parts, that is, the solutions are of the form $x = \pm x_(y)$, $u = u_*(y)$.*

Proof. We will prove that $b = \sqrt{a^2 - x_0^2}$, where $x_0 = \pi/(2a)$. Note that the pair $x = x_0$, $y = b$ satisfies the system (12) with $u = a$, $k = 0$.

(i) If $Q(x + iu, y) = 0$ and $|y| > b$, $0 \leq u \leq a$, then $x^2 = u^2 - y^2 < a^2 - b^2 = x_0^2$. Hence $|xu| < x_0 a = \pi/2$, and the equation $xu = \pi(2k + 1)$ cannot hold.

(ii) Let $\gamma = \pi/2$ or $-\pi/2$. Substituting $x = \gamma/u$ into the equation $x^2 + y^2 = u^2$, we get

$$u^4 - y^2 u^2 - \gamma^2 = 0.$$

Since u is real, there must be $u^2 \geq 0$, hence

$$u^2 = \frac{1}{2} \left(y^2 + \sqrt{y^2 + 4\gamma^2} \right). \quad (18)$$

Let $u_*(y)$ be the (positive) square root of the RHS. Clearly, $u_*(y)$ is an increasing function. By definition of the number b (at the beginning of the proof), we have $u_*(b) = a$. Hence, $|y| \leq b$ implies $0 \leq u_* < a$, as required.

If we replace the value $|\gamma| = \pi/2$ by $\pi(k+1/2)$ with $k \geq 1$, then the three inequalities $u^2 \geq \gamma^2$, $0 \leq u \leq a$, and $a^2 < 3\pi/2 \leq |\gamma|$, are incompatible.

The rest is trivial. The value $\gamma = \pi/2$ yields the solution $u = u_*$, $x = x_* = \pi/(2u_*)$, while $\gamma = -\pi/2$ yields the solution $u = u_*$, $x = -x_*$. \square

Lemma 7 *There exist real numbers $b > 0$ and a_1, a_2 satisfying the inequalities*

$$\sqrt{\frac{\pi}{2}} < a_1 < a_2 < \sqrt{\frac{3\pi}{2}}, \quad (19)$$

with the following properties:

- (i) the equation $Q(z, y) = 0$ does not have solutions (z, y) with $y \in \mathbb{R}$, $|y| \geq b$, $0 \leq \operatorname{Im} z \leq a_1$;
- (ii) for every $y \in (-b, b)$, the system

$$Q(z, y) = 0, \quad 0 \leq \operatorname{Im} z < a_2$$

has exactly two solutions $z_{\pm}(y) = \pm x_*(y) + iu_*(y)$.

Proof. Choose a_1 and a_2 satisfying the inequalities (19) arbitrarily. Take any $a \in (a_1, a_2)$ and determine b as in Lemma 6. Let us check that the required properties are in place.

(i) If $|y| > b$, then by Lemma 6, (i), the equation $Q(z, y) = 0$ does not have solutions with $0 \leq \operatorname{Im} z < a$. The constraint $\operatorname{Im} z < a_1$ is even stronger.

(ii) Let $|y| < b$. By Lemma 6, (ii), the equation $Q(z, y) = 0$ has two solutions $z = \pm x_* + iu_*$ satisfying the inequality $0 \leq \operatorname{Im} z < a$. Relaxing the constraint to $0 \leq \operatorname{Im} z < a_2$ could potentially bring in extra solutions. For every such solution, $u = \operatorname{Im} z$ would be defined by (18) with $|\gamma| = \pi(k + 1/2)$, $k \geq 1$. Then $u^2 \geq 3/2\pi$, which contradicts the condition $u < a_2 < \sqrt{3\pi/2}$. \square

Lemma 8 *Let a_1, a_2, b be as in Lemma 7 and I_1, I_2 as in (16). Then*

$$\begin{aligned} I_1 &= O(e^{-\lambda a_1}), \\ I_2 &= -\pi i \sum_{\pm} \int_{-b}^b \frac{e^{i\lambda z_{\pm}(y)}}{z_{\pm}(y)} dy + O(e^{-\lambda a_2}), \end{aligned} \tag{20}$$

where $z_{\pm}(y)$ are the two solutions of the equation $Q(z, y) = 0$ defined in Lemma 7.

Proof. 1) Consider the integral (17) with $|y| > b$. By Lemma 7 (i) and (14) where we set $h = a_1$,

$$T(y, \lambda) = \int_{-\infty}^{\infty} \frac{e^{\lambda(-a_1+ix)}}{Q(x+ia_1, y)} dx.$$

Therefore,

$$|I_1| \leq \int_{|y|>b} |T(y, \lambda)| dy \leq e^{-\lambda a_1} \int_{|y|>b} \int_{-\infty}^{\infty} \frac{dx dy}{|Q(x+ia_1, y)|}.$$

The conditions of Lemma 5 are met with $u = a_1$, $K = (-\infty, -b] \cup [b, \infty)$. The estimate (20) for I_1 follows at once.

2) In (14) we set $h = a_2$. By Lemma 7 (ii) there are two poles, $z = z_{\pm}(y)$, that contribute to the sum of residues. The residue at z is

$$\frac{e^{i\lambda z}}{Q'_z(z, y)} = \frac{e^{i\lambda z}}{2ze^{z^2+y^2}} = \frac{e^{i\lambda z}}{-2z},$$

since at a pole $Q(z, y) = 0$ and $e^{z^2+y^2} = -1$. We have identified the sum part of the RHS in (14) with the first part in the RHS of the second estimate in (20).

To show that the integral in the RHS of (14) is $O(e^{-\lambda a_2})$, it suffices to repeat the same argument as in part 1 of this proof with obvious changes: $u = a_2$, $K = [-b, b]$. \square

To simplify the remaining calculations, note that

$$z_+ = x_* + iu_*, \quad z_- = -x_* + iu_*,$$

so

$$iz_- = \overline{iz_+}.$$

Therefore

$$-\pi i \sum_{\pm} \frac{e^{i\lambda z_{\pm}}}{z_{\pm}} = 2\pi \operatorname{Re} \frac{e^{i\lambda z_+}}{iz_+}. \quad (21)$$

Lemma 9

$$\int_{-b}^b \frac{e^{i\lambda z_+(y)}}{iz_+(y)} \sim -2^{1/2} \pi^{1/4} e^{-\lambda \sqrt{\pi/2}} e^{i(\lambda \sqrt{\pi/2} + \pi/8)} \quad (22)$$

as $\lambda \rightarrow +\infty$.

Proof. The function $\operatorname{Re}(iz_+(y)) = -u_*(y)$ attains its maximum $-\sqrt{\pi/2}$ at $y = 0$, cf. (18).

The saddle point method tells us that

$$\int_{-b}^b \frac{e^{i\lambda z_+(y)}}{iz_+(y)} \sim \frac{e^{i\lambda z_+(0)}}{iz_+(0)} \cdot \left(\frac{2\pi}{-i\lambda z_+''(0)} \right)^{1/2}.$$

Recalling that $z_+^2 = \pi i - y^2$, evaluate:

$$iz_+(0) = i\sqrt{\pi} e^{i\pi/4} = \sqrt{\frac{\pi}{2}} (-1 + i)$$

(since $\text{Re} z_+, \text{Im} z_+ > 0$).

Expanding $z_+(y)$ in powers of y^2 , we find

$$z_+(y) = (\pi i)^{1/2} \left(1 - \frac{y^2}{2\pi i} \right) + O(y^4),$$

hence

$$z_+''(0) = \sqrt{\pi} e^{i\pi/4} \frac{1}{-\pi i}$$

and

$$-i z_+''(0) = \frac{1}{\sqrt{\pi}} e^{i\pi/4}.$$

Thus

$$\int_{-b}^b \frac{e^{i\lambda z_+(y)}}{i z_+(y)} \sim \frac{e^{(-1+i)\lambda\sqrt{\pi/2}}}{\sqrt{\pi} e^{3\pi i/4}} \left(\frac{2\pi^{3/2}}{\lambda e^{3\pi i/4}} \right)^{1/2}.$$

Simplifying, we obtain (22). □

Combining (20), (21), (22), we get

$$I_2 \sim -2^{3/2} \pi^{5/4} e^{-\lambda\sqrt{\pi/2}} \frac{\cos(\lambda\sqrt{\pi/2} + \pi/8)}{\lambda^{1/2}},$$

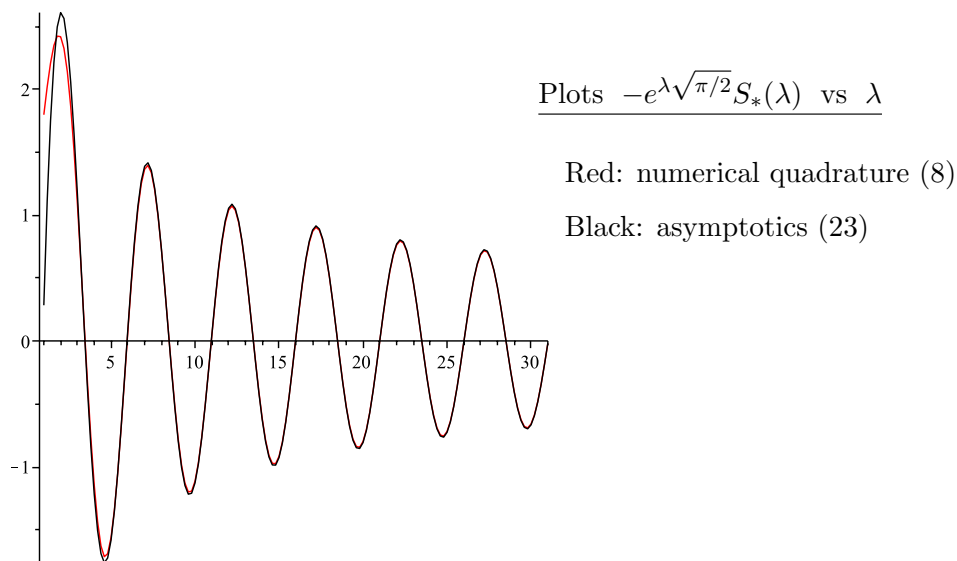
and, using (16),

$$S_*(\lambda) \sim 2^{3/2} \pi^{1/4} e^{-\lambda\sqrt{\pi/2}} \frac{\cos(\lambda\sqrt{\pi/2} + \pi/8)}{\lambda^{1/2}}. \quad (23)$$

The final answer in the original notation, cf. (7), is

$$S(t) \sim 2\pi^{1/4} e^{-\sqrt{2\pi t}} \frac{\cos(\sqrt{2\pi t} + \pi/8)}{t^{1/4}}.$$

(24)



Remark. From our analysis it is easy to see that the (unwritten) error term in (23) is

$$O\left(e^{-\lambda\sqrt{\pi/2}}\lambda^{-3/2}\right).$$

The form in which the answer is presented in (23) or (24), is slightly inaccurate: the error term needs not be dominated by the main term *everywhere*, since the latter becomes zero for some values of λ (or t).

The meaning of the \sim sign is that the error term is smaller by its *order of magnitude*, which is characterized by the *non-oscillating* (amplitude) factor in (23) or (24). Regarding the usage of the \sim sign in this and similar situations, I disagree with de Bruijn who expressed negative opinion of such usage (*Asymptotic methods in analysis*, end of Sect. 5.11).